

# Constructive derivation of asynchronicity

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## Abstract

Under the Lorentz transformation, when two inertial frames synchronize their clocks at one space-time point, their clocks at other spatial locations display an asynchronization given by the formula  $-vx\gamma/c^2$ . This asynchronization term is distinctive in that it transforms a set of instantaneous events in one frame to a set of non-instantaneous events in the other frame leading to the principle of ‘relativity of simultaneity.’ In this paper we provide a constructive kinematical derivation of this asynchronization from the point of view of any one inertial frame.

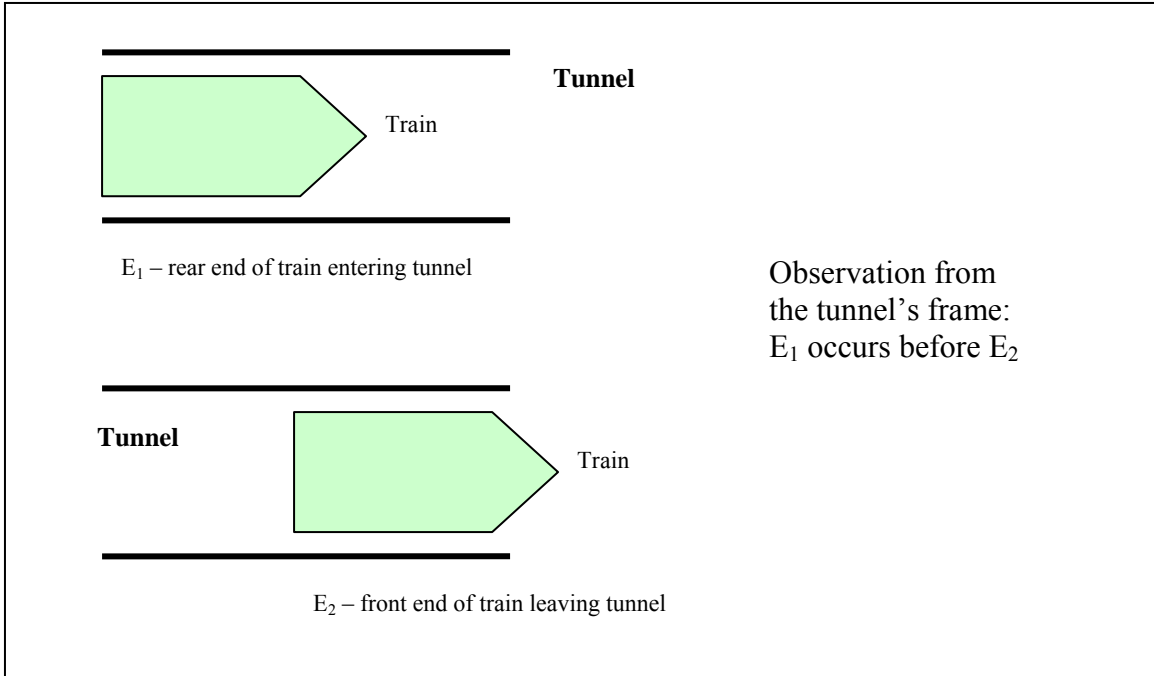
## I. Introduction

Einstein [1] showed that the Lorentz transformation equations describe the relation between  $(x, y, z, t)$  and  $(x', y', z', t')$  under the principles of equivalence of inertial frames and constancy of the speed of light. The Lorentz transformation equations, according to the principle of relativity, describe the relationship between event coordinates observed by inertial frames in uniform relative motion. Under these transformations an object of proper length  $L_0$ , appears to have contracted to a length  $L$  as seen from another inertial frame; the relationship between  $L$  and  $L_0$  is given by the formula  $L = L_0 / \gamma$ , where  $\gamma = \frac{1}{\sqrt{1 - v^2 / c^2}}$  and  $c$  is the speed of light in vacuum.

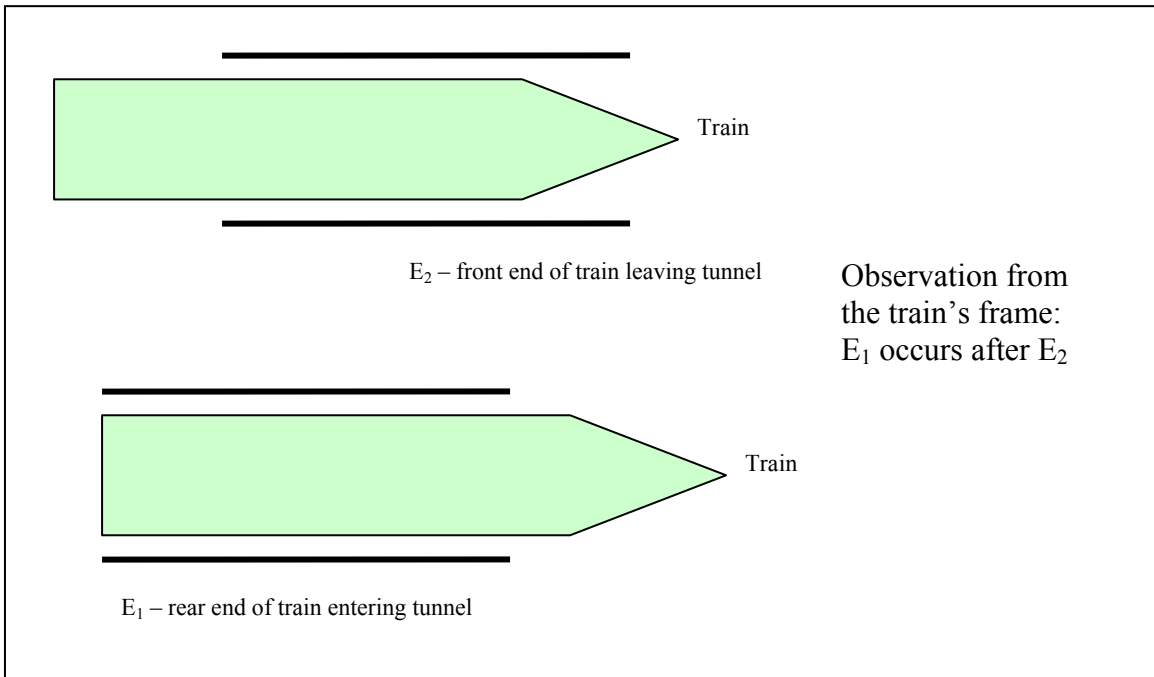
The transformation for the time coordinate according to the Lorentz transformation is  $t' = (t - vx/c^2) \gamma$ .

The Lorentz transformations can be derived with a preferred inertial frame [2] as well as under the principle of equivalence of all inertial frames [1]. According to [2], one need not accept Lorentz’s philosophy of a preferred frame of reference to accept a Lorentzian pedagogy.

Students who are embarking on their first study of Special Relativity are normally confounded by mutual contraction of length as well as mutual slowing down of time. Each of these appears contradictory and a source of all kinds of paradoxes such as the rod–slot paradox, the train–tunnel paradox, and a few others [3, 4, 5]. The observation of mutual length contraction is a result of a mismatch in the perception of an instant. In the simplest form of the train–tunnel paradox (shown in Figures 1 and 2), the two events – the rear end of the train entering the tunnel (event  $E_1$ ) and the front end of the train leaving the tunnel (event  $E_2$ ) are reversed in time order, as observed from the frames of the train and the tunnel. In the tunnel’s frame  $E_1$  happens before  $E_2$ , thus the train’s length is fully within the tunnel during the time interval  $E_2 \sim E_1$ . In the train’s frame,  $E_2$  happens before  $E_1$  and thus the train was never fully in the tunnel, which is ‘due to the length of the train being greater than the length of the tunnel.’ Figures 1 and 2 illustrate how relative lengths are observed, not by actual measurement of lengths, but by measurement of the time-order of certain events. So, in general, all length paradoxes reduce to a reversal in time-order



**Figure 1. Observation from the tunnel's frame**



**Figure 2. Observation from the train's frame**

of two events as observed by observers in a pair of inertial frames. The principle of relativity of simultaneity states that 'two events  $E_1$  and  $E_2$  which are simultaneous and spatially separated in an inertial frame will not be simultaneous in any other inertial frame.' If we take  $E_1$  as the space-time origin in both the frames, then the differences in the time interval between  $E_1$  and  $E_2$  is

basically the asynchronicity between the frames. Therefore any constructive methodology of understanding asynchronicity leads to a clearer understanding of the principle of relativity of simultaneity and the theory of Special Relativity. In this paper we show that the asynchronicity between inertial frames is directly linked to the ‘apparent slowing down of clocks in a moving frame.’ Thus in our approach, four concepts are integrated in understanding the relationship between the running of clocks and asynchronicity. These concepts are (i) running of clocks, (ii) asynchronicity, (iii) relativity of simultaneity, and (iv) relativity in length observations. This approach is useful in understanding Special Relativity, even if we adopt the principle of “equivalence of inertial frames” for further developments in Physics [2].

When we consider the instant  $t = 0$  in a ‘stationary’ frame  $K$ , clocks in a ‘moving’ frame  $M$  show a time of  $(-vx/c^2)\gamma$ . Thus the instant  $t = 0$  in frame  $K$  translates into a set of non-instantaneous events in frame  $M$  with  $t' = (-vx/c^2)\gamma$ . This term is called the phase difference [6, 7] and can be taken to be a property of the Lorentz transformation. In this context, the principle of relativity of simultaneity can be restated as a principle of relativity of synchronicity as follows: Spatially separated and synchronized clocks of any inertial frame appear asynchronous from any other inertial frame.

A constructive explanation of asynchronicity is given in text books [6, 7]. Such an explanation envisages setting up two clocks, A and B, equidistant from the spatial origin and synchronizing them with a light ray emanating from the origin [6, pp 71-73]. The process as it happens in frame  $M$  is observed by frame  $K$ . If the distance between the origin and the clocks is ‘ $s$ ,’ then as the light ray reaches the clocks A and B in frame  $M$ , frame  $K$  observes that these clocks have moved a distance  $sv/(c + v)$  and  $sv/(c - v)$ ; the movement of the two clocks is unequal and also towards and away from the approaching light ray. Thus the difference in distance the light ray had to travel to meet clock A in comparison to clock B is the sum of these two terms which is  $2svc/(c^2 - v^2)$ , and after taking into consideration the apparent length contraction effect, the phase difference can be accounted for.

The reason for the phase difference term (or asynchronization) in this pedagogy is that while observers in frame  $M$  assumed that the two clocks A and B were stationary, observers in frame  $K$  observed that clocks A and B moved towards and away from the approaching light ray during the time interval the light ray traveled from the spatial origin of  $M$  to clocks A and B. Since for both frames  $K$  and  $M$  the speed of light is the same, frame  $K$  considers that the clocks in frame  $M$  were not synchronized properly, as the movement of the clocks A and B were not accounted for by frame  $M$ . This same process is considered by frame  $M$  as valid, and therefore frames  $K$  and  $M$  do not agree on the synchronicity of each other’s clocks. This pedagogy is based on the principle of the constancy of speed of light in all inertial frames.

In this paper we offer an alternative purely constructive pedagogy for this phase difference or asynchronicity. It is well accepted that an observer in a given inertial frame relies and trusts his own observations as he has no other options or that is his best option. In such a context, by assuming only that moving clocks run or appear to run as a function of their velocity, the determination of the asynchronicity between the frames is derived succinctly by using the definition of the derivative. Also the assumption of an arbitrary function  $f(v)$  (instead of the actual function  $1/\gamma$ ) makes the mathematics simpler rather than complex. Later, by substituting the actual function for  $f(v)$  the resultant asynchronicity is shown to be in agreement with the Lorentz transformations.

We also show that in the context of the symmetry between the frames there are two possibilities for  $f(v)$  and the application of the existence of an upper limit for speed reduces it to one function, namely  $1/\gamma$ , which is also in accordance with the Lorentz transformations.

## II. Synchronization of spatially separated clocks

We start with an inertial frame  $K$ , which is our reference frame. There can be no preferred direction in this inertial frame. Therefore, all directions are equivalent or, in other words, frame  $K$  is isotropic. In frame  $K$ , suppose a set of identical clocks is placed at the same location and at rest with respect to each other; their identicalness is easily verifiable by comparing them periodically. Furthermore it is also a simple task to synchronize them so that they all show the same time always. According to the Lorentz transformations, if any clock moves at a uniform velocity, it will (appear to) run slow. As frame  $K$  will go by its own observations as real, as far as  $K$  is concerned, moving clocks run slow and we assume that they (appear to) run slow by a factor  $f(v)$  which is a function of the uniform velocity  $v$  of the moving clock. We note that by definition  $f(0) = 1$  and since frame  $K$  is isotropic,  $f(+v) = f(-v)$  and consequently  $f'(0) = 0$ .

Now suppose we wish to separate these clocks to different spatial locations  $x_1, x_2, x_3, \dots$ , etc. In general, a particular clock is moved slowly to a location  $x$  at a velocity  $v$  and then it is left at location  $x$ . In this process, the time taken will be  $(x/v)$  in frame  $K$ , while the clock that is being moved will show an elapsed time of  $(x/v) f(v)$ . Hence, after reaching location  $x$ , the clock would have developed an asynchronicity of  $(x/v) (f(v) - 1)$ . It is evident that  $f(0) = 1$ . Therefore we can rewrite the asynchronicity as  $(x/v) (f(v) - f(0))$ . If we separate the clock very slowly and take this proposition to the mathematical limit, we can let  $v \rightarrow 0$  and then obtain the asynchronicity to be  $= x f'(0)$ .

Since frame  $K$  is isotropic,  $f(v)$  is an even function (i.e.,  $f(+v) = f(-v)$ ) and  $f'(0) = 0$ . Hence the asynchronicity  $x f'(0) = 0$ . Therefore the method of slowly separating clocks yields a set of spatially separated yet synchronized clocks in an isotropic stationary frame. Now consider a moving frame  $M$ , moving with a velocity  $V$  with respect to frame  $K$ . Assume that an observer in frame  $M$  considers  $M$  to be stationary and isotropic and applies the same method for synchronizing spatially separated clocks as done by an observer in frame  $K$ .

**Observers in frame  $K$  would record their view of the slow separation process in frame  $M$  as follows:** One of the clocks in frame  $M$  is separated by some distance  $s$  from  $M$ 's spatial origin; this is achieved by increasing the velocity  $V$  of one of the clocks at  $M$ 's spatial origin to  $(V+v)$ . Thus the absolute velocity of the moving clock<sup>1</sup> is  $V+v$  as seen by frame  $K$ . The time taken for the separation<sup>2</sup> is  $s/v$ , where  $s$  is the displacement between  $M$ 's spatial origin (where the set of identical clocks are located), and the final position of the separated clock. In this time interval 'stationary' clocks in frame  $M$  indicate a time interval of  $(s/v) f(V)$  and the moving clock that is undergoing the separation in frame  $M$  runs up a time  $(s/v) f(V+v)$ . At the end of the separation the asynchronicity developed between the two clocks in frame  $M$ , the one at  $M$ 's spatial origin and the one at a distance  $s$  from  $M$ 's spatial origin will be  $[(s/v) f(V+v)] - [(s/v) f(V)]$ . Setting the limit  $v \rightarrow 0$  as before, we get the asynchronicity to be  $= sf'(V)$ . In general, this asynchronization

<sup>1</sup> Within frame  $M$  the velocity of the moving clock may be observed to be different from  $v$ ; this does not concern observers in frame  $K$ .

<sup>2</sup> According to observers in frame  $K$ , in an elapsed time of  $(s/v)$ , 'stationary' clocks in frame  $M$  travel a distance  $(s/v)V$  and the 'moving' clock in frame  $M$  travels a distance  $(s/v)(V+v)$ , thus achieving between them a separation of  $s$ .

is zero only when  $V = 0$  (i.e., for frame  $K$ ). However frame  $M$  believes that the spatially separated clocks are synchronous because frame  $M$  is believed to be stationary and isotropic.

Thus it is observed that an isotropic frame ( $K$ ) achieves a set of spatially separated yet synchronous clocks by slow separation of clocks. If moving clocks slow down in their running by a factor  $f(V)$ , then when a moving frame adopts the same procedure of slow separation for creating a set of spatially separated yet synchronized clocks, the moving frame develops a set of spatially separated but asynchronous clocks; frame  $M$  believes them to be synchronous and, as we will show, believes that the clocks of frame  $K$  are asynchronous because frame  $K$  is moving with respect to frame  $M$ . We may also note that if we assume the clocks of frame  $M$  to be synchronous, then frame  $M$  is (virtually) isotropic. In Section III below, we derive the function  $f(V)$  by assuming symmetry of the space-time coordinate transformations between the two inertial frames  $K$  and  $M$ .

### III. Selecting the function $f(V)$ to achieve symmetry between frames $K$ and $M$

Consider a clock in frame  $K$  located at a distance  $x$  from the origin and showing a time  $t$ . A clock in the moving frame  $M$  located at this same space-time point shows a time  $t'$ , which is a function of  $(x, t)$ , where  $x$  is the location of the two clocks, which are momentarily together (one in  $K$  and the other co-moving with  $M$ ) at the instant  $t$  in frame  $K$ . We denote this function as  $F(x, t) = t'$ , where  $t'$  is the time shown by the clock in frame  $M$ . From the discussions in Section II we get Equation (1) below.

$$F(x, 0) = x f'(V) \quad (1)$$

Also, at  $t = 0$ ,  $F(0, 0) = 0$ , corresponding to the clocks at the origins of frames  $K$  and  $M$ . When the clock at the origin of frame  $M$ , moves to the location  $x = Vt$ , the time shown by a clock in frame  $K$  at that location is  $t$ , because the velocity of any object in frame  $M$  is  $V$  as observed by frame  $K$ . Since clocks in frame  $M$  run at a rate  $f(V)$  times the rate of clocks in frame  $K$  (as observed by observers in frame  $K$ ), the time shown by the clock at the origin of frame  $M$  will be  $t f(V)$ , when it reaches the location  $x = Vt$ . Therefore,

$$F(Vt, t) = t f(V) \quad (2)$$

A clock in frame  $M$  located at  $(x, 0)$  and showing a time  $x f'(V)$  according to equation (1) will, after an elapsed time of  $t$  in frame  $K$ , move to a location  $(x + Vt)$  and add to itself a time duration  $t f(V)$ , giving rise to the equation

$$F(x + Vt, t) = x f'(V) + t f(V) \quad (3)$$

Without any loss of generality, Equation (3) can be written as

$$F(x, t) = (x - Vt) f'(V) + t f(V) \quad (4)$$

With some rearranging, Equation (4) can be rewritten as

$$F(x, t) = x f'(V) + t [f(V) - V f'(V)] \quad (5)$$

We place a requirement that clocks in frame  $K$ , appear to run at the rate  $f(V)$  as observed from frame  $M$ . The clock at the origin of frame  $K$  ( $x = 0$ ), after an elapsed time of  $t$ , will show a time  $t$ .

The clock in frame  $M$  at that location will read  $F(0, t)$ . Since frame  $M$  observes any particular clock in frame  $K$  to be running at the rate  $f(V)$  times the clocks in frame  $M$ , we have  $t = F(0, t) f(V)$ . Thus we get the equation

$$F(0, t) = t \frac{1}{f(V)} \quad (6)$$

Comparing equations (5) and (6) and setting  $x$  to zero in equation (5), we obtain

$$t \frac{1}{f(V)} = t [f(V) - V f'(V)]$$

Or

$$\frac{1}{f(V)} = f(V) - V f'(V) \quad (7)$$

Solving the differential equation (7) by separation of variables, we obtain two solutions

$$f(V) = \sqrt{1 - k^2 V^2} \quad \text{when } f(V) \leq 1, \quad \text{and} \quad f(V) = \sqrt{1 + k^2 V^2} \quad \text{when } f(V) > 1.$$

For the moment we ignore the solution where  $f(V) > 1$  and select the solution  $f(V) = \sqrt{1 - k^2 V^2}$  as this alone will lead to the constancy of the speed of light. The second solution corresponding to  $f(V) > 1$ , can be easily shown to be same as the rotation of the orthogonal axis in planar rotation.

At the same time it can also be a model when moving clocks run faster by a factor  $\sqrt{1 + k^2 V^2}$ .

Therefore, if moving clocks (appear to) slow down according to the function  $f(V) = \sqrt{1 - k^2 V^2}$ , then the viewpoint from frames  $K$  and  $M$  will be symmetric in that both will see each other's clocks as slowing down.

This function  $f(V) = \sqrt{1 - k^2 V^2}$  has a mathematical limitation that  $k^2 V^2$  must be less than 1, as otherwise it would become imaginary. So it automatically sets a limit on the speed as  $1/k$ . Thus by setting  $1/k$  as the maximum observed speed, which is the speed of light ( $c$ ), we get the time dilation factor to be  $\sqrt{1 - V^2/c^2}$  and this is in accordance with the Lorentz transformations.

Furthermore, when we set  $f(V) = \sqrt{1 - V^2/c^2}$  and evaluate the expression  $x f'(V)$ , which is the asynchronicity, its value turns out to be  $-Vxy/c^2$ . This value for the asynchronicity is also in full agreement with the Lorentz transformations.

#### IV. Summary

We were able to derive the time dilation function by assuming a simple procedure for clock synchronization (slow separation of clocks) and apparent symmetry between the frames. Under the principle of equivalence of the inertial frames, 'moving clocks run slow' is not the preferred statement. The preferred statement should be that 'moving clocks appear to run slow' because all

clocks in all inertial frames are equivalent and they appear to run slow with respect to each other. However, from a given inertial frame  $K$ , an observer would like to understand why clocks in another inertial frame are not synchronous with his clocks and why the asynchronicity proportionally increases with distance. The approach described here clearly shows by a constructive methodology that under a uniform principle or method of synchronizing clocks in both the frames, the clocks in both the frames become asynchronous. Standard texts such as [7] have shown this by assuming the slowing down functions *a priori* to be  $1/\gamma$  and further assuming second order terms are negligible (under complex algebra with binomial expansion and fractional indices). In this presentation no such assumptions are made and yet the derivation turns out to be quite simple under an arbitrary function  $f(V)$  when the definition of the derivative is invoked. Further it is shown that this asynchronicity is in agreement with the asynchronicity obtained from the Lorentz transformations when we impose the two conditions, namely symmetry between the frames and an upper limit for speed.

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## Figure Captions

Figure 1. Observation from the tunnel's frame

Figure 2. Observation from the train's frame