

Relative Motion and the Speed of Light

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Abstract

Considering relative motion in two-dimensional Euclidean space, a constructive derivation is given for the one-way speed of light in a moving frame in relative motion with respect to another frame presumed to be at rest. The one-way speed is shown to be equal to c under a preferred resynchronization in the moving frame. Even if the moving frame uses the preferred synchronization convention of the presumed rest frame, the average round-trip speed remains c on any closed path in the moving frame.

1 Introduction

The view point [1, 2, 3] that the Lorentz length contraction and time dilation are sufficient to explain the round-trip speed of light being a constant [4] has been discussed widely in the literature. The alternative view is that the length contraction and time dilation are only ‘apparent’ observations and since the inertial frames are equivalent, there was in actuality no contraction [5, p. 60]. A discussion on the necessity or otherwise of adopting a frame-specific synchronization is given in [6, 7, 8, 9].

In this paper we give a constructive derivation for the one-way speed of light in a moving frame based on Lorentz length contraction and time dilation. We also show that the same variable one-way speed of light becomes a constant when the reference frame adopts the Einsteinian synchronization convention [6] as applicable to itself.

2 Derivation of One-Way speed of Light

In Figure 1, AB is the path of a light ray in the presumed ‘rest’ frame (K). A'B' is the apparent path of the light ray in a moving frame, K', which is moving at a velocity v with respect to K. We assume universal simultaneity of events. Under this scheme the Einsteinian synchronicity convention of frame K is adopted as the synchronicity convention by all frames [1]. Thus events that are simultaneous in K are also simultaneous in K'. The resultant transformation has been named the IST transformation [1] and its components are (i) movement, (ii) length contraction, and (iii) time-dilation [9]. When we combine the preferred synchronization convention of K' [5, 6] to the IST transformation [1], we obtain the Lorentz transformation [9]. For the discussion in Sections 2 and 3 we use the IST transformation, and in Section 4 we switch to the preferred synchronization convention of K'.

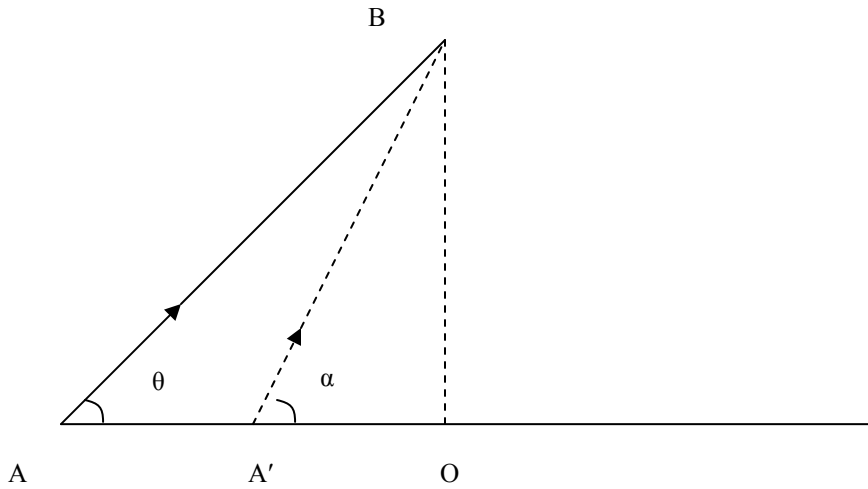


Figure 1

One may note that as observed by frame K:

$ \begin{aligned} AB &= ct \\ BO &= ct \sin\theta \\ AO &= ct \cos\theta \\ AA' &= vt \\ A'O &= [ct \cos\theta - vt] \end{aligned} $
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$$A'O = \gamma[ct \cos\theta - vt] \rightarrow \text{as observed by } K' \text{ after taking into account the Lorentz contraction}$$

Thus for frame K' :

$$BO = ct \sin\theta \tag{1a}$$

$$A'O = \gamma[ct \cos\theta - vt] \tag{1b}$$

Here one should note that $A'O$ did not contract but appeared to expand. This is because $A'O$ is a distance between two points measured as $[ct \cos\theta - vt]$ in frame K . The same distance is measured by frame K' using contracted rulers and hence the distance appears longer. In general an object in K' will contract along the line of motion, but distances measured by K' using contracted rulers will appear longer.

$$\begin{aligned}
 \tan\alpha &= (BO/A'O) = ct \sin\theta / \{ \gamma [ct \cos\theta - vt] \} \\
 &= \sin\theta / \{ \gamma [\cos\theta - (v/c)] \}
 \end{aligned}$$

that is,
$$\tan \alpha = \frac{\sin \theta}{\gamma(\cos \theta - (v/c))} \quad (2)$$

$$\begin{aligned} \tan^2 \alpha &= \frac{\sin^2 \theta}{\gamma^2 \left(\cos \theta - \frac{v}{c} \right)^2} \\ &= \frac{1 - \cos^2 \theta}{\gamma^2 \left(\cos \theta - \frac{v}{c} \right)^2} \end{aligned}$$

Cross multiplying and grouping terms, we get

$$\left(1 + \gamma^2 \tan^2 \alpha\right) \cos^2 \theta - 2\gamma^2 \frac{v}{c} \tan^2 \alpha \cos \theta + \left(\frac{\gamma^2 v^2}{c^2} \tan^2 \alpha - 1\right) = 0$$

Solving for $\cos \theta$ by the standard quadratic method, we obtain

$$\cos \theta = \frac{\gamma^2 \frac{v}{c} \tan^2 \alpha \pm \sec \alpha}{\left(1 + \gamma^2 \tan^2 \alpha\right)} \quad (3)$$

Now the apparent speed of light in K' will be

$$C^+ = \frac{A'B}{(t/\gamma)} \quad (4)$$

where $A'B$ is to be measured by K' as $(A'O^2 + BO^2)^{1/2}$ and in the denominator we use (t/γ) due to Lorentz time dilation. One may also note that the rulers of K' contract along line $A'B$ in such a way that the equation $A'B = (A'O^2 + BO^2)^{1/2}$ is maintained. This means that the contraction of rulers along different directions in the $x'y'$ plane happens in such a way that the Euclidean norm represented by the Pythagorean theorem is maintained.

In terms of the rulers of K' , in the right angle triangle $A'BO$, by considering equations (1a) and (1b), we have

$$A'B = ct \sqrt{(\sin \theta)^2 + \left[\gamma \left(\cos \theta - \frac{v}{c}\right)\right]^2}$$

And substituting the above into equation (4), we obtain

$$C^+ = c\gamma \sqrt{\sin^2 \theta + \left[\gamma \left(\cos \theta - \frac{v}{c}\right)\right]^2} \quad (5)$$

Dividing both numerator and denominator by γ^2 in equation (3), and simplifying, we get

$$\begin{aligned}
\cos \theta &= \frac{\frac{v}{c} \tan^2 \alpha \pm \sec \alpha \left(1 - \frac{v^2}{c^2}\right)}{1 - \frac{v^2}{c^2} + \tan^2 \alpha} \times \frac{\cos^2 \alpha}{\cos^2 \alpha} \\
&= \frac{\frac{v}{c} \sin^2 \alpha \pm \cos \alpha \left(1 - \frac{v^2}{c^2}\right)}{1 - \frac{v^2}{c^2} \cos^2 \alpha} \\
\cos \theta - \frac{v}{c} &= \frac{\frac{v}{c} \sin^2 \alpha \pm \cos \alpha \left(1 - \frac{v^2}{c^2}\right) - \frac{v}{c} + \frac{v^3}{c^3} \cos^2 \alpha}{1 - \frac{v^2}{c^2} \cos^2 \alpha} \\
&= \frac{-\frac{v}{c} (1 - \sin^2 \alpha) + \frac{v^3}{c^3} \cos^2 \alpha \pm \cos \alpha \left(1 - \frac{v^2}{c^2}\right)}{1 - \frac{v^2}{c^2} \cos^2 \alpha} \\
&= \frac{-\frac{v}{c} \cos^2 \alpha \left(1 - \frac{v^2}{c^2}\right) \pm \cos \alpha \left(1 - \frac{v^2}{c^2}\right)}{1 - \frac{v^2}{c^2} \cos^2 \alpha} \\
&= \frac{\cos \alpha \left(1 - \frac{v^2}{c^2}\right) \left[-\frac{v}{c} \cos \alpha \pm 1\right]}{1 - \frac{v^2}{c^2} \cos^2 \alpha} \\
&= \frac{\frac{\cos \alpha}{\gamma^2}}{1 + \frac{v}{c} \cos \alpha} \quad \text{OR} \quad \frac{\frac{-\cos \alpha}{\gamma^2}}{1 - \frac{v}{c} \cos \alpha} \\
(\cos \theta - v/c)\gamma &= \frac{\left(\frac{\cos \alpha}{\gamma}\right)}{1 + \frac{v}{c} \cos \alpha} \quad \text{OR} \quad \frac{\left(\frac{-\cos \alpha}{\gamma}\right)}{1 - \frac{v}{c} \cos \alpha}
\end{aligned}$$

For the case, $\theta = 0$, α is also 0, and substituting these values, we find that the first solution is the appropriate one. For the case $\theta = \pi$, α is also π and again we find that the first solution is the appropriate one.

Therefore,

$$(\cos \theta - v/c)\gamma = \frac{\left(\frac{\cos \alpha}{\gamma}\right)}{1 + \frac{v}{c} \cos \alpha} \quad (6)$$

$$\text{Equation (2) can be rewritten as } (\cos \theta - v/c)\gamma = \frac{\sin \theta}{\tan \alpha} \quad (2a)$$

Comparing equations (6) and (2a), the left hand sides are equal and therefore equating the right hand sides we obtain,

$$\sin \theta = \frac{\left(\frac{\sin \alpha}{\gamma}\right)}{1 + \frac{v}{c} \cos \alpha} \quad (7)$$

Substituting for $(\cos \theta - v/c)\gamma$ from equation (6) and for $\sin \theta$ from equation (7) into equation (5)

we get

$$C^+ = \frac{c}{1 + \frac{v}{c} \cos \alpha} \quad (8)$$

With the above formula for the one-way speed of light, the average round-trip speed of light along any path is c , as explained in the next section.

3 Light reflected by a series of mirrors

Consider a light ray that gets reflected by a series of mirrors but either returns to the starting point or to a point where the x' coordinate is same as that of the starting point. Then the average speed observed by frame K' of that light ray will be c , even though it traveled at different speeds between mirrors i and j , that is $v_{i,j} \neq c$. But the average speed $v_{\text{avg}} = c$.

In Figure 2, the light ray can be visualized as traveling along the curve by a series of reflecting mirrors closely placed along the path taken by the curve, that is, an arbitrary polygon with one side as the y' -axis and other sides approximating the curve as much as possible. The starting point is P and the ending point is Q, so that

$\int_P^Q dx = 0$. For a light ray tracing the path along the sides of the polygons as described, the time taken in each line segment can be expressed as

$$\Delta t = \frac{\sqrt{\Delta x^2 + \Delta y^2}}{c} \text{ using the right hand side of equation (8).}$$

$$\left(1 + \frac{v}{c} \cos \alpha\right)$$

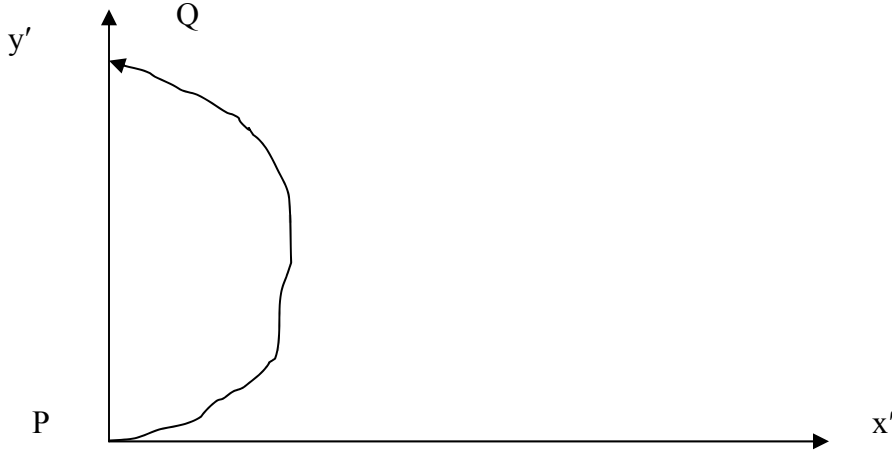


Figure 2

The above expression for Δt can be re-written as

$$\Delta t = \frac{\sqrt{\Delta x^2 + \Delta y^2}}{c} \left(1 + \frac{v}{c} \cos \alpha\right)$$

The quantity $\cos \alpha$ can be approximated to $\frac{1}{\sqrt{1 + \frac{\Delta y^2}{\Delta x^2}}}$. Substituting this for $\cos \alpha$ and upon

further simplification, we obtain

$$\Delta t = \frac{\sqrt{\Delta x^2 + \Delta y^2}}{c} + \frac{v}{c^2} \Delta x \tag{9}$$

The summation of the first term on the right hand side of equation (9) gives a time interval as if light traveled the entire path at a speed of c . The summation of the second term on the right hand side of equation (9) is 0 because $\sum \Delta x = 0$.

4 Resynchronization

With the speed of light as given in equation (8), any instance of the propagating light ray in the moving frame will be given by

$$(x, y, t) = \left(\frac{ct \cos \alpha}{1 + \frac{v}{c} \cos \alpha}, \frac{ct \sin \alpha}{1 + \frac{v}{c} \cos \alpha}, t \right) \quad (10)$$

Here, without loss of generality, we use (x, y, t) for the moving frame. Resynchronizing along the x -axis by subtraction proportional to the x coordinate by a factor v/c^2 we obtain the following [1, 9]:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -v/c^2 & 0 & 1 \end{pmatrix} \begin{pmatrix} ct \cos \alpha / \{1 + [(v/c) \cos \alpha]\} \\ ct \sin \alpha / \{1 + [(v/c) \cos \alpha]\} \\ t \end{pmatrix} = \begin{pmatrix} x \\ y \\ T \end{pmatrix}$$

Note that the space coordinates remain unaltered in value; hence we have retained the xy representation for the space coordinates. The time coordinate is altered under the re-synchronization as below

$$T = t - \frac{v}{c^2} \left(\frac{ct \cos \alpha}{1 + \frac{v}{c} \cos \alpha} \right)$$

$$\text{Or } t = \frac{T}{1 - \frac{(v/c) \cos \alpha}{1 + (v/c) \cos \alpha}}$$

$$t = T [1 + (v/c) \cos \alpha] \quad (11)$$

Substituting for t in equation (10) we get

$$x = cT \cos \alpha$$

$$y = cT \sin \alpha$$

Therefore with the resynchronized time $T = \frac{t}{1 + (v/c) \cos \alpha}$, the moving frame observes that light travels at the constant speed of c in all directions on the xy plane.

5 Summary

In a two-dimensional spatial system, a light ray propagating at an angle θ with respect to the x -axis, forms an event at any given time t as $(ct \cos \theta, ct \sin \theta, t)$. This event is transformed under IST transformation to $[(ct \cos \theta - vt)\gamma, ct \sin \theta, t/\gamma] = (x', y', t'_I)$ and under the Lorentz

transformation, the same event becomes $(x', y', \left[\frac{t}{\gamma} - \frac{v}{c^2} x' \right]) = (x', y', t'_E)$. In both cases the

observed angle of propagation becomes $\alpha = \tan^{-1}(y'/x')$ and evidently remains the same for the IST and Lorentz transformed coordinates. By solving for $\cos \theta$ in terms of α and substituting we obtained the event coordinates for the case of the IST transformation to be

$$(C^+ t'_I \cos \alpha, C^+ t'_I \sin \alpha, t'_I) \text{ where } C^+ = \frac{c}{1 + (v/c) \cos \alpha}.$$

For the case of the Lorentz transformation, we obtained the event coordinates to be $(ct'_E \cos \alpha, ct'_E \sin \alpha, t'_E)$. The one-way speed of light is constant under the Lorentz transformation and the average round-trip speed of light is constant in the case of the IST transformation. Since the Michelson–Morley experiment determined that the average round-trip speed of light is constant, both the IST as well as the Lorentz transformations conform to the result of the Michelson–Morley experiment.

6 Conclusions

We constructively derived the one-way speed of light in a moving frame (equation 8). We have shown that it becomes equal to c under a preferred resynchronization [6] in the moving frame. Even under the preferred synchronization convention of another inertial frame, the average round-trip speed remains equal to c on any closed path in the moving frame.

References

- [1] de Abreu, R. and Guerra, V. (2008). The principle of relativity and the indeterminacy of special relativity. *Eur. J. Phys.* **29** pp.33-52.
- [2] Brown, H and Pooley, O. (2001). The origins of the spacetime metric: Bell’s Lorentzian pedagogy and its significance in general relativity. *Physics meets Philosophy at the Planck Scale* ed C Callendar and N Huggett (Cambridge: Cambridge University Press) pp 256-72.
- [3] Bell, J. (1976). How to teach special relativity. *Prog. Sci. Cult.* **1**, pp. 67-80.
- [4] Michelson, A. and Morley, E. (1887). On the relative motion of the earth and the luminiferous ether. *Am. J. Sci.* **134**, p. 133.
- [5] Einstein, A. (1961). *Relativity, The Special and General Theory*, Authorized Translation by Robert W. Lawson, Three Rivers Press, New York, p. 60.
- [6] Ohanian, H. (2004). The role of dynamics in the synchronization problem. *Am. J. of Phys.* **72** (2), pp. 141-148.
- [7] Macdonald, A. (2005). Comment on ‘The role of dynamics in the synchronization problem,’ by H. Ohanian, *Am. J. Phys.* **72** (2), 141-148 (2004). *Am. J. Phys.* **73** (5), pp. 454-455.
- [8] Martinez, A. (2005). Conventions and inertial reference frames. *Am. J. Phys.* **73** (5), pp. 452-454.
- [9] Iyer, C. (2008). Comment on ‘The principle of relativity and the indeterminacy of special relativity,’ by de Abreu and Guerra. *Eur. J. Phys.* **29**, pp. L13-L17.